

# INITIAL SCHEMES OF VERY AFFINE SEVERI VARIETIES

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**ABSTRACT.** Severi varieties are parameter spaces whose points correspond to nodal curves on toric surfaces. We study their initial schemes, which are certain flat degenerations. We find an explicit combinatorial description of them in terms of subdivisions of polygons. It provides an understanding of the tropicalizations of Severi varieties and enables us to do tropical intersections to compute the degrees of Severi varieties.

## 1. INTRODUCTION

The advent of tropical geometry and its fast development suggest to look at the classical algebraic geometry in a different perspective.

Tropicalization is an operation that turns subvarieties of an algebraic torus into polyhedral objects in a real space (the real part of the Lie algebra of the torus) along with a locally-constant integral-valued function on it. This procedure enables us to build an intersection theory on the algebraic torus called *tropical intersection theory* which can be used to solve classical enumerative questions. There are many attractive properties of this intersection theory. First of all, we deal with polyhedral objects instead of algebraic varieties. Also we often do not need to consider compactifications.

The case of hypersurfaces is very related to Newton polytope theory. The tropicalization of a hypersurface defined by a Laurent polynomial  $f$  is the codimension 1 skeleton of the outer-normal fan of  $\text{Newton}(f)$ , the Newton polytope of  $f$ . Moreover, the integral-valued function on it is determined by the lattice lengths of the edges of  $\text{Newton}(f)$ .

In general, however, for a subvariety of higher codimension it is not easy to understand its tropicalization, unless we have enough information about its defining ideal which provides descriptions of initial ideals.

In this paper, we study the initial schemes of very affine Severi varieties. Severi varieties are projective varieties which parameterize nodal curves on toric surfaces. Our main geometric objects are the intersections of Severi varieties with the big open torus of the ambient projective space, which are called *very affine* Severi varieties. The results obtained on them provide an explicit description of a part of the tropicalizations of Severi varieties.

In the forthcoming paper, we will show that this part is enough to compute the degrees of Severi varieties using tropical intersection theory.

We heavily rely on Shustin's works ([Shustin1],[Shustin2]) on the tropicalizations and patchworking theories of nodal curves on a toric surface. Roughly speaking, he found a nice combinatorial characterization of the tropicalizations of nodal curves in terms of subdivisions of a polygon. To obtain such a nice characterization, however, he imposed the base-point condition, that is, he considered the nodal curves which pass through a certain number of points in a general position.

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Our main idea is that we separate the base-point-condition from his arguments and understand the enumerations of nodal curves in terms of products in the ring of tropical varieties (§2.4)

The main theorem is presented in Theorem 4.1. Roughly speaking, for a nice weight order  $\omega$ , the initial scheme  $\text{in}_\omega \text{Sev}(\Delta, \delta)$  of a very affine Severi variety  $\text{Sev}(\Delta, \delta)$  is a union of finitely many translates of a subtorus of the big open torus. The number of such translates can also be computed by a purely combinatorial formula.

Section 2 contains preliminaries which in particular introduces a numeric invariant  $\text{rank}(\omega)$  of the weight order  $\omega$  and this invariant replaces Shustin's base-point-condition. In section 3, we study several parameter spaces of curves. They are much simpler than Severi varieties and given by explicit systems of binomial equations. They are building blocks to study the initial schemes of very affine Severi varieties. In section 4, we review Shustin's works which are used to prove the main theorem and then we prove the main theorem.

## 2. PRELIMINARIES

By a *scheme* we shall mean an algebraic scheme over the complex field  $\mathbb{C}$ , that is, a scheme  $X$  together with a morphism of finite type from  $X$  to  $\text{Spec}(\mathbb{C})$ . A *variety* will be a reduced scheme, and a *subvariety* of a scheme will be a closed reduced subscheme. A *point* on a scheme will always be a closed point. A *curve* is a 1-dimensional scheme. In this paper, we consider only subschemes of the algebraic torus  $\mathbb{T} = \text{Spec}(\mathbb{C}[\mathbb{Z}^n])$ . The subvarieties of  $\mathbb{T}$  are called *very affine varieties*. The notations and restrictions given here are used throughout the paper.

**2.1. Initial Schemes.** We review on the initial schemes of an affine scheme. We only consider the subschemes of the algebraic torus  $\mathbb{T} = \text{Spec}(\mathbb{C}[\mathbb{Z}^n]) = \text{Spec}(\mathbb{C}[x_1^\pm, \dots, x_n^\pm])$ .

Let  $\omega \in \mathbb{Z}^n$  be an integral vector. We think of  $\omega$  as a function on monomials:

$$\omega(x^a) := \omega \cdot a, \text{ where } x^a = x_1^{a_1} \cdots x_n^{a_n}. \quad (2.1)$$

Then  $\omega$  gives a partial order  $>_\omega$  on the monomials:  $x^a >_\omega x^b \Leftrightarrow \omega(x^a) > \omega(x^b)$ .

Given  $g \in \mathbb{C}[\mathbb{Z}^n]$ , we write  $\text{in}_\omega g$  for the sum of all the terms of  $g$  that are maximal with respect to the partial order  $>_\omega$ . If  $I$  is an ideal in  $\mathbb{C}[\mathbb{Z}^n]$  we write  $\text{in}_\omega I$  for the ideal in  $\mathbb{C}[\mathbb{Z}^n]$  generated by  $\text{in}_\omega g$  for all  $g \in I$ , which is called *the initial ideal* of  $I$  with respect to  $\omega$ . The subscheme  $V(\text{in}_\omega I) := \text{Spec}(\mathbb{C}[\mathbb{Z}^n]/\text{in}_\omega I)$  defined by  $\text{in}_\omega I$  is called *the initial scheme* of  $V(I) := \text{Spec}(\mathbb{C}[\mathbb{Z}^n]/I)$  with respect to  $\omega$ .

We may consider  $V(\text{in}_\omega I)$  as a flat degeneration of  $V(I)$ , that is, there exist a one-parameter flat family  $V(I_t)$  such that  $V(I) = V(I_1)$  and  $V(\text{in}_\omega I) = V(I_0)$ . The details are given in [Eisenbud, §15.8].

The following is the case that we often consider in this paper.

Let  $X$  be a subset of the algebraic torus  $\mathbb{T}$ . Let  $I(X)$  be the *radical* ideal of  $X$  in  $\mathbb{C}[\mathbb{Z}^n]$ , that is,

$$I(X) := \{f \in \mathbb{C}[\mathbb{Z}^n] : f(p) = 0 \text{ for all } p \in X\}. \quad (2.2)$$

Let  $\bar{X}$  be the very affine variety defined by  $I(X) = I(\bar{X})$ . We always assume that  $X$  is open dense in  $\bar{X}$ . For simplicity, we write  $\text{in}_\omega X$  for the initial scheme of the very affine variety  $\bar{X}$  with respect to  $\omega$ , which is called *the initial scheme* of  $X$ .

The following proposition provides a description of the initial schemes which is crucial for our study of Severi varieties. We consider an extension of the complex field: Let  $\mathbb{K}$  denote the field of locally convergent Puiseux series over  $\mathbb{C}$ , that is, the elements of  $\mathbb{K}$  are power series of the form

$$b(t) = \sum_{\tau \in R} c_\tau t^\tau, \quad (2.3)$$

where  $R \subset \mathbb{Q}$  is contained in an arithmetic progression bounded from above and  $\sum_{\tau \in R} |c_\tau| t^\tau < \infty$  for sufficiently large positive  $t$ . This is an algebraically closed field of characteristic zero with a non-Archimedean valuation

$$\text{Val}(b) = \max\{\tau \in R : c_\tau \neq 0\}. \quad (2.4)$$

Without loss of generality we may suppose that  $\text{Val}(b)$  is an integer by changing the parameter  $t \mapsto t^l$  for some  $l$ . We always assume this unless mentioned otherwise. (Note that this definition of  $\mathbb{K}$  is slightly different from the one in [Shustin2]. However, the one in [Shustin2] can be obtained by the substitution,  $t \mapsto t^{-1}$ .)

**Proposition 2.1.** *The set of closed points of  $\text{in}_\omega X$  equals to*

$$\{\bar{z} \in \mathbb{T} : \text{there exists } z = \bar{z}t^\omega + \text{l.o.t.} \in X(\mathbb{K}) \subset \mathbb{T}(\mathbb{K})\}, \quad (2.5)$$

where l.o.t. stands for “lower order terms” and  $z$  is in vector-notation.

*Proof.* The proof for the inclusion  $\subset$  is presented in [Kazarnovskii1, Proposition 4]. and a more general version when  $X$  is a subvariety of  $(\mathbb{K} \setminus \{0\})^n$  is presented in [Speyer, Lemma 2.1.3.]

Let us consider the other inclusion  $\supset$ . Suppose  $z = \bar{z}t^\omega + \text{l.o.t.} \in X(\mathbb{K})$  and let  $f \in I(X)$ . It is enough to show that  $\text{in}_\omega f(\bar{z}) = 0$ , which follows from the fact that  $\text{in}_\omega f(\bar{z})$  is the constant term of  $f(z) \cdot t^{-m} \in \mathbb{C}[t]$ , where  $m$  is the  $\omega$ -degree of  $f$ .  $\square$

**2.2. Very Affine Severi Varieties.** In this section, we define our main geometric object, the very affine Severi varieties (of curves on toric surfaces).

Let  $\Delta \subset \mathbb{R}^2$  be a 2-dimensional lattice polygon. (In this paper, every polytope is assumed to be a convex lattice polytope.) We fix a vertex of  $\Delta$  (called the base index of  $\Delta$ ) and assume that it is at the origin  $(0, 0) \in \mathbb{R}^2$ . Let  $X_\Delta$  denote the projective toric surface constructed from  $\Delta$ . It comes with the tautological linear system  $\mathbb{P}(\mathcal{L}_\Delta)$ . The linear system  $\mathbb{P}(\mathcal{L}_\Delta)$  is the parameter space of plane curves whose defining polynomials have Newton polygon contained in  $\Delta$ . It is a projective space of dimension  $|\Delta_{\mathbb{Z}}| - 1$ , where  $\Delta_{\mathbb{Z}} = \Delta \cap \mathbb{Z}^2$  is the set of lattice points in  $\Delta$ . Let  $\mathbb{T}_\Delta$  denote the big open torus of  $\mathbb{P}(\mathcal{L}_\Delta)$ . It parameterizes plane curves whose defining polynomials have no missing monomial, that is, the polynomials have a non-zero coefficient for the term corresponding to every lattice point in  $\Delta$ . In particular, their Newton polygons are always *equal* to  $\Delta$ . By assuming that the constant term is always 1, we can identify a curve parameterized by a point in  $\mathbb{T}_\Delta$  with its defining polynomial. Also such curve uniquely determines its closure in the toric surface  $X_\Delta$ .

Thus, in this paper, we don't distinguish a point in  $\mathbb{T}_\Delta$  from the curve in  $(\mathbb{C} \setminus \{0\})^2$ , the curve in the toric surface  $X_\Delta$  or the defining polynomial corresponding to the point.

**Definition 2.2.** Given a non-negative integer  $\delta \leq |\text{Int}(\Delta) \cap \mathbb{Z}^2|$ , we consider the set  $X$  consisting of  $f \in \mathbb{T}_\Delta$  such that  $f$  defines a curve in  $(\mathbb{C} \setminus \{0\})^2$  having exactly  $\delta$  nodes (that is, ordinary double points) as its only singularities. Let  $\text{Sev}(\Delta, \delta) \subset \mathbb{T}_\Delta$  denote the closure of  $X$  in  $\mathbb{T}_\Delta$ . It is called a *very affine Severi variety*. ( $X$  is open dense in  $\text{Sev}(\Delta, \delta)$ .)

Note that we obtain the classical Severi variety by taking the closure of  $\text{Sev}(\Delta, \delta)$  in  $\mathbb{P}(\mathcal{L}_\Delta)$  and it is of pure dimension  $|\Delta_{\mathbb{Z}}| - 1 - \delta$ . We impose one restriction on the polygon  $\Delta$  to have  $\dim(\text{Sev}(\Delta, \delta)) = |\Delta_{\mathbb{Z}}| - 1 - \delta$ , that is,  $\text{Sev}(\Delta, \delta)$  is dense in the classical Severi variety. This restriction is satisfied in most cases.

**2.3. Subdivisions of  $\Delta$  and Adjacency graph.** The main theorem 4.1 provides descriptions of the initial schemes of  $\text{Sev}(\Delta, \delta)$  in terms of subdivisions of the polygon  $\Delta$ , which is very combinatorial. In this section, we review some terminologies and define a numeric invariant of subdivisions of  $\Delta$  which is crucial for our main theorem.

**Definitions 2.3.**

- (1) A *subdivision* of  $\Delta$ , denoted by  $\mathcal{S}(\Delta)$ , is a decomposition of  $\Delta$  into a finite number of non-degenerate sub-polygons such that the intersection of any two of these sub-polygons is a common face of both of them (maybe empty). (We consider  $\Delta$  as a subdivision of  $\Delta$  with one 2-dimensional face.)
- (2) Given a subdivision  $\mathcal{S}(\Delta)$ , let  $\text{Vertices}(\mathcal{S}(\Delta))$ ,  $\text{Edges}(\mathcal{S}(\Delta))$ ,  $\text{Faces}(\mathcal{S}(\Delta))$ ,  $\text{Triangles}(\mathcal{S}(\Delta))$ ,  $\text{Parallelograms}(\mathcal{S}(\Delta))$ ,  $\text{Int}(\mathcal{S}(\Delta)) \cap \mathbb{Z}^2$  be the set of vertices, edges, (2-dimensional) faces, triangles, parallelograms, interior lattice points of  $\mathcal{S}(\Delta)$ , respectively.
- (3) A subdivision  $\mathcal{S}(\Delta)$  is called
  - *triangular* if every 2-dimensional face is a triangle;
  - *nodal* if every 2-dimensional face is either a triangle or a parallelogram;
  - *simple* if every lattice point on the boundary of  $\Delta$  is a vertex of  $\mathcal{S}(\Delta)$ .
- (4) A subdivision  $\mathcal{S}(\Delta)$  is called *regular* if there exists a continuous concave piecewise-linear function on  $\Delta$  whose domains of linearity are precisely the 2-dimensional faces of  $\mathcal{S}(\Delta)$ .

Now let  $\psi : \Delta_{\mathbb{Z}} \rightarrow \mathbb{R}$  be a real-valued function defined on  $\Delta_{\mathbb{Z}}$ , the set of all lattice points on  $\Delta$ . We construct a regular subdivision  $\Delta_{\psi}$  of  $\Delta$  from  $\psi$  as follows:

Let  $G_{\psi} \subset \mathbb{R}^3$  be the convex hull of the set

$$\{(a, y) : y \leq \psi(a), \quad a \in \Delta_{\mathbb{Z}}\}. \quad (2.6)$$

Then the upper boundary of  $G_{\psi}$  is the graph of a concave piecewise-linear function  $cc_{\psi}$  which is called the *concave hull* of  $\psi$ . (The upper boundary of  $G_{\psi}$  is by definition the union of faces of  $G_{\psi}$  which do not contain vertical half-lines.) Let  $\Delta_{\psi}$  denote the regular subdivision of  $\Delta$  given by the domains of linearity of  $cc_{\psi}$ .

Let  $C(\psi)$  be the set of all continuous *concave* piece-wise linear functions on  $\Delta$  whose domains of linearity induce the subdivision  $\Delta_{\psi}$ . As embedded in  $\mathbb{R}^{\Delta_{\mathbb{Z}}}$ ,  $C(\psi)$  is a polyhedral cone in  $\mathbb{R}^{\Delta_{\mathbb{Z}}}$  with  $cc_{\psi}|_{\Delta_{\mathbb{Z}}} \in C(\psi)$ .

Notice that  $cc_{\psi}|_{\Delta_{\mathbb{Z}}}$  may not coincide with  $\psi$  and in such case  $\psi \notin C(\psi)$ .

**Definition 2.4.** By definition, the *rank* of  $\psi$ , written as  $rank(\psi)$ , is the dimension of the cone  $C(\psi)$ .

**Proposition 2.5.** Suppose that  $\mathcal{S}(\Delta) = \Delta_{\psi}$  is a regular nodal subdivision of  $\Delta$ . Then,

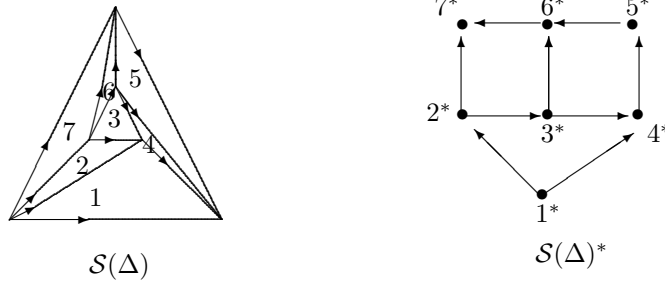
$$rank(\psi) = |Vertices(\mathcal{S}(\Delta))| - 1 - |Parallelograms(\mathcal{S}(\Delta))|. \quad (2.7)$$

The proof of this Proposition can be easily deduced from [IMS, Lemma 2.40.]

*Remark 2.6.* The regular subdivisions of a lattice polytope is studied in [GKZ, Ch.7] in which the Secondary fan is introduced. A close connection of the cone  $C(\psi)$  and  $rank(\psi)$  to the Secondary fan will be studied in the forthcoming paper.

Now we study the adjacency graph  $\mathcal{S}(\Delta)^*$  of a given subdivision  $\mathcal{S}(\Delta)$  of  $\Delta$ . By definition, the vertices  $F^*$  of  $\mathcal{S}(\Delta)^*$  correspond to the 2-dimensional faces  $F$  of  $\mathcal{S}(\Delta)$  and two vertices  $F_1^*, F_2^*$  of  $\mathcal{S}(\Delta)^*$  are connected by an edge  $(F_1^*, F_2^*)$  if the corresponding faces  $F_1, F_2$  of  $\mathcal{S}(\Delta)$  have a common edge  $(F_1, F_2)$  in  $\mathcal{S}(\Delta)$ .

Given an orientation  $\Gamma$  on  $Edges(\mathcal{S}(\Delta))$ , we define an orientation  $\Gamma^*$  on  $Edges(\mathcal{S}(\Delta)^*)$  as follows: Direct  $F_1^* \rightarrow F_2^*$ , if the oriented edge  $(F_1, F_2)$  and a normal vector to it leaving from  $F_1$  to  $F_2$  are positively oriented.



Observing the duality between  $\Gamma$  and  $\Gamma^*$ , it is straightforward to see the following:

- $\Gamma$  has an oriented cycle if and only if  $\Gamma^*$  has either a sink or a source of valency at least 3. (A sink (resp. source) is a vertex  $v$  such that all edges adjacent to  $v$  are coming into (resp. leaving from)  $v$ .)
- $\Gamma^*$  has an oriented cycle if and only if  $\Gamma$  has either a sink or a source of valency at least 3.

For any subdivision  $\mathcal{S}(\Delta)$  of  $\Delta$ , we can always find an orientation  $\Gamma$  on  $Edges(\mathcal{S}(\Delta))$  such that  $\Gamma$  has neither oriented cycle, sink, nor source. For example, choose a generic vector  $\zeta \in \mathbb{R}^2 \setminus \{0\}$  and orient the edges of  $\mathcal{S}(\Delta)$  so that they form acute angles with  $\zeta$ .

**2.4. The ring of tropical varieties.** Tropical geometry has been being developed very actively and it became a very rich and broad field in Mathematics. In this section, we review some basics of tropical geometry which are used to present our main results. Refer to [BJSST, Kazarnovskii1, Kazarnovskii2, Sturmfels-Tevelev] for more details.

**Definitions 2.7.**

- (1) A  $d$ -dimensional *tropical variety* (or *homogeneous tropical variety of degree  $n - d$* ) is a pair  $(\mathcal{T}, \mathbf{m})$ , where  $\mathcal{T}$  is a subset of  $\mathbb{R}^n$  and  $\mathbf{m} : \mathcal{T}^\circ \rightarrow \mathbb{Z}_{>0}$  is a locally constant function, called *intrinsic multiplicity* which satisfies:
  - There exists a pure  $d$ -dimensional rational polyhedral fan supported on  $\mathcal{T}$ ;
  - $\mathcal{T}^\circ \subset \mathcal{T}$  is the open subset of regular points, where  $\omega \in \mathcal{T}$  is called *regular* if there exists a vector subspace  $L_\omega \subset \mathbb{R}^n$  such that  $\mathcal{T} = L_\omega$  locally near  $\omega$ .
  - The function  $\mathbf{m}$  satisfies the balancing condition (see Definition 2.11) for one (and hence for any) fan supported on the set  $\mathcal{T}$ .
- (2) A *tropical variety* is a formal sum of homogeneous tropical varieties of different degrees.

Tropical varieties form a graded commutative algebra  $\mathbb{A}$  as defined in [Kazarnovskii2]. To a subvariety  $X$  of the algebraic torus  $\mathbb{T} = \text{Spec}(\mathbb{C}[\mathbb{Z}^n])$ , we can assign an element  $\text{Trop}(X)$  of  $\mathbb{A}$  called the *tropicalization* of  $X$  (See Definition 2.8). This correspondence determines an intersection theory of subvarieties of  $\mathbb{T}$ . In fact,  $\mathbb{A}$  is a special case of the ring of conditions of a complete symmetric variety introduced by [DP].

We summarize this correspondence: Let  $X_1$ , and  $X_2$  be subvarieties of  $\mathbb{T}$ .

- $\text{Trop}(X_1) = \text{Trop}(X_2)$  if and only if  $d = \dim(X_1) = \dim(X_2)$  and for every subvariety  $Y \subset \mathbb{T}$  of dimension  $n - d$ , the following holds: the number of points in the sets  $X_1 \cap gY$  and  $X_2 \cap gY$  is the same for all  $g \notin D(X_1, X_2, Y)$ , where  $D(X_1, X_2, Y)$  is some proper subvariety of  $\mathbb{T}$ .
- $\text{Trop}(X_1 \cup X_2) = \text{Trop}(X_1) + \text{Trop}(X_2)$
- $\text{Trop}(X_1 \cap gX_2) = \text{Trop}(X_1)\text{Trop}(X_2)$  for all  $g \notin D(X_1, X_2)$ , where  $D(X_1, X_2)$  is a proper subvariety of  $\mathbb{T}$ .
- If  $X_1$  and  $X_2$  are of complimentary dimensional, then  $\text{Trop}(X_1)\text{Trop}(X_2)$  is the origin in  $\mathbb{R}^n$  with intrinsic multiplicity equal to the number of intersection points in  $X_1 \cap gX_2$  for a generic  $g \in \mathbb{T}$ . We identify  $\text{Trop}(X_1)\text{Trop}(X_2)$  with this non-negative integer. For example,
  - the product of  $n$  tropicalizations of hypersurfaces in  $\mathbb{T}$  is the mixed volume of the Newton polyhedra of the hypersurfaces times  $n!$ ; (compare [D. Bernstein].)
  - let  $\mathbb{T}_1$ , and  $\mathbb{T}_2$  be two subtori of  $\mathbb{T}$  of complimentary dimension. Then their tropicalizations are rational linear subspaces of  $\mathbb{R}^n$  with constant multiplicity 1 and they are of complimentary dimensional. The product  $\text{Trop}(\mathbb{T}_1)\text{Trop}(\mathbb{T}_2)$  equals to the (normalized) volume of the parallelepiped defined by the fundamental cells of the lattices  $\text{Trop}(\mathbb{T}_1) \cap \mathbb{Z}^n$ , and  $\text{Trop}(\mathbb{T}_2) \cap \mathbb{Z}^n$ .

**Definition 2.8.** Let  $X$  be a subvariety of the algebraic torus  $\mathbb{T} = \text{Spec}(\mathbb{C}[\mathbb{Z}^n])$ .

- (1) A vector  $\omega = (\omega_1, \dots, \omega_n)$  in  $\mathbb{Z}^n$  is called a *c-vector* of  $X$  if it satisfies either of the followings:
  - (algebraic)  $\text{in}_\omega I(X)$  contains no monomial, where  $I(X)$  is the ideal of  $X$  in  $\mathbb{C}[\mathbb{Z}^n]$ .
  - (geometric) there exists a germ of a curve  $\rho : \mathbb{C}^* \rightarrow \mathbb{T}$ , such that
    - $\rho$  is defined near  $\infty$ , that is, defined on  $\{u \in \mathbb{C}^* \mid |u| > r\}$  for sufficiently large  $r$ ;
    - $\rho$  is given by a Laurent series

$$\rho(t) = (a_1 t^{\omega_1} + l.o.t., \dots, a_n t^{\omega_n} + l.o.t.), \quad (2.8)$$

where  $a_i \in (\mathbb{C} \setminus \{0\})$ ,  $(i = 1, \dots, n)$  and *l.o.t.* means “lower order terms”;

- The image of  $\rho$  is contained in  $X$ .
- (2) The (support of) tropicalization of  $X$  denoted by  $\text{Trop}(X)$  is the closure of the union of positive rays  $\mathbb{R}_{\geq 0} \cdot \omega$  generated by all *c-vectors*  $\omega$  of  $X$ .
  - (3) We define the *intrinsic multiplicity*  $\mathbf{m}(\omega)$  of a point  $\omega$  in  $\text{Trop}(X)^\circ$  to be the sum of the multiplicities of all minimal associate primes of the initial ideal  $\text{in}_\omega I(X)$ .

*Remark 2.9.*

- The algebraic and the geometric definitions of *c-vector* given above are equivalent to each other by the so-called Fundamental Theorem of tropical geometry. ([Speyer, Theorem 2.1.2])
- The  $c$  in front of *c-vector* stands for current, adapted from Kazarnovskii’s *c-fan* ([Kazarnovskii2]).
- The  $\text{Trop}(X)$  has the structure of a tropical variety with dimension equal to  $\dim(X)$ .
- The fact that the intrinsic multiplicity  $\mathbf{m}(\omega)$  defined as above satisfies the balancing condition is proved in [Speyer, §2.5].

Geometrically, for  $\omega$  a regular point of  $\text{Trop}(X)$ , the initial scheme  $\text{in}_\omega X$  is a union of finitely many translates of a subtorus  $\mathbb{T}'$  (connected closed subgroup) of  $\mathbb{T}$ . And  $\mathbf{m}(\omega)$  is the number of such translates counted with multiplicity. As elements of the algebra  $\mathbb{A}$ , we may write

$$\text{Trop}(\text{in}_\omega X) = \mathbf{m}(\omega) \cdot \text{Trop}(\mathbb{T}'). \quad (2.9)$$

**Notation 2.10.** We fix the following notation related to toric varieties. Let  $\mathbb{T} = \mathbb{T}_{\mathbf{k}}$  denote an algebraic torus over a field  $\mathbf{k}$ . We write  $M$  for the lattice of characters of  $\mathbb{T}$  and  $N := M^\vee$  for the lattice of one parameter subgroups. The tropical variety  $\text{Trop}(X)$  lives in  $N_{\mathbb{R}}$ . For any fan  $\mathcal{F} \subset N_{\mathbb{R}}$ , we denote by  $\Sigma(\mathcal{F})$  the corresponding toric variety and by  $X(\Sigma)$  the closure of  $X$  in  $\Sigma(\mathcal{F})$ .

**Definition 2.11** ([Fulton-Sturmfels]). Given a rational polyhedral cone  $\sigma \subset N_{\mathbb{R}}$ , we write  $N_\sigma$  for the sublattice of  $N$  generated by  $\sigma \cap N$ . For a facet  $\tau$  of  $\sigma$  let  $n_{\sigma,\tau}$  be any representative in  $\sigma$  for the generator of the 1-dimensional lattice  $N_\sigma/N_\tau$ . Let  $\mathbf{m}$  be an integer-valued function on the set of  $k$ -dimensional cones of some fan  $\mathcal{F}$ . Then  $\mathbf{m}$  is said to satisfy the *balancing condition* if, for any cone  $\tau$  of  $\mathcal{F}$  of dimension  $k-1$ , we have

$$\sum_{\sigma \supset \tau} \mathbf{m}(\sigma) \cdot n_{\sigma,\tau} \in N_\tau, \quad (2.10)$$

where the sum is over all  $k$ -dimensional cones  $\sigma$  in  $\mathcal{F}$  that contain  $\tau$ .

### 3. CLOSED SUBGROUPS OF $\mathbb{T}_\Delta$

In this section we study several subvarieties of  $\mathbb{T}_\Delta$ . They are all parameter spaces whose points correspond to curves in toric surface  $X_\Delta$  with certain properties. They have a role of building blocks for the study of the Sever variety  $\text{Sev}(\Delta, \delta)$ . In fact, I believe that they are fundamental objects in the ring of tropical varieties of  $\mathbb{T}_\Delta$ . They are very simple in a sense that all of them are closed subgroups of  $\mathbb{T}_\Delta$  up to translations. (Let  $\mathbb{G}$  be a subgroup of  $\mathbb{T}_\Delta$  and  $g \in \mathbb{T}_\Delta$ . We call  $g\mathbb{G}$  a translate of  $\mathbb{G}$ .) Therefore, their tropicalizations are all real linear spaces with constant multiplicities given by the number of their components. In fact, all of the subvarieties we consider are special cases of the last one presented in §3.6. However, each one of them is worthy to present separately: the proofs are much simpler than the one for the general case and provide explicit constructions.

#### 3.1. $\mathbb{V}_{\partial\Delta} \subset \mathbb{T}_\Delta$ .

**Definition 3.1.** Let  $\mathbb{V}_{\partial\Delta}$  denote the set of all  $f \in \mathbb{T}_\Delta$  such that the restriction of  $f$  on each side of  $\Delta$  is a pure power of a binomial (up to multiplication by a monomial).

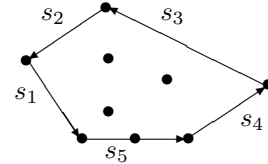
*Remark 3.2.* Geometrically, points of  $\mathbb{V}_{\partial\Delta}$  correspond to curves in the toric surface  $X_\Delta$  such that they cross the union of the toric divisors at precisely  $l$  points at which they are unibranch, where  $l$  is the number of sides of  $\Delta$ .

**Theorem 3.3.**

- (1)  $\mathbb{V}_{\partial\Delta}$  is a closed subgroup of  $\mathbb{T}_\Delta$  up to a translation.
- (2) Its dimension equals to  $|\text{Edges}(\Delta)| - 1 + |\text{Int}(\Delta) \cap \mathbb{Z}^2|$ .
- (3) Its number of components is  $l(\partial\Delta)$ , where  $l(\partial\Delta)$  is the greatest common divisor of the lattice lengths of sides of  $\Delta$ .

*Proof.* This is a special case of [Khovanskii].

Assign the counter-clockwise orientation on the sides of  $\Delta$  and order them as  $s_1, \dots, s_m$  such that the head of  $s_1$  is the origin (i.e., the base index) and the tail of  $s_1$  equals to the head of  $s_2$  and similarly for all the other sides.



Let  $f \in \mathbb{V}_{\partial\Delta}$ . The restriction of  $f$ ,  $f_{s_i}$ , on each side  $s_i$ , is quasi-homogeneous and thus it can be identified with a univariate polynomial such that the head and the tail of  $s_i$  correspond to the leading term and the constant term, respectively. Then  $f_{s_i}$  has a unique multiple root  $\xi_i$  and the coefficient of the leading term of  $f_{s_1}$  equals to 1. Therefore  $f$  restricted on the sides of  $\Delta$  is uniquely determined by  $\xi_1, \dots, \xi_m$  with one following relation:

$$(-\xi_1)^{|s_1|} \dots (-\xi_m)^{|s_m|} = 1, \quad (3.1)$$

where  $|s_i|$  is the lattice length of  $s_i$ , ( $i = 1, \dots, m$ ).

Notice that  $f$  can have any non-zero complex number for every interior lattice point.

Thus, the statements follow in straightforward.

□

3.2.  $\mathbb{V}_{\partial\Delta, \mathbb{P}^1} \subset \mathbb{V}_{\partial\Delta}$ .

**Definition 3.4.** Let  $\mathbb{V}_{\partial\Delta, \mathbb{P}^1}$  denote the set of all  $f \in \mathbb{V}_{\partial\Delta}$  such that  $f$  defines a rational curve.

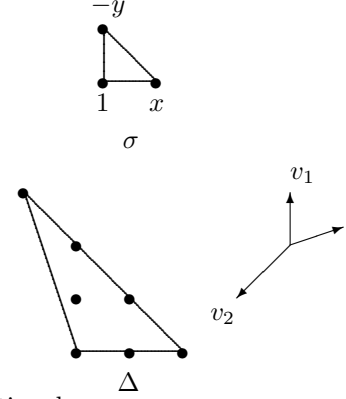
**Theorem 3.5.** If  $\Delta$  is a triangle, then  $\mathbb{V}_{\partial\Delta, \mathbb{P}^1}$  is a 2-dimensional subtorus of  $\mathbb{T}_\Delta$  up to a translation.

*Proof.*

Let  $\sigma$  be the standard 2-simplex, that is, the convex hull of the points,  $(0, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ . Let  $f_\sigma = x - y + 1$ . Then  $f_\sigma$  is in  $\mathbb{V}_{\partial\sigma, \mathbb{P}^1}$  with the rational parametrization:

$$\phi_\sigma : \theta \mapsto (x(\theta) = \theta, y(\theta) = \theta + 1) \quad (3.2)$$

Let  $\Psi_\Delta : (x, y) \mapsto (x^{s_1 v_{11}} y^{s_2 v_{21}}, x^{s_1 v_{12}} y^{s_2 v_{22}})$  be the toric morphism corresponding to the linear transformation of  $\mathbb{R}^2$  given by the matrix  $\begin{pmatrix} s_1 v_{11} & s_2 v_{21} \\ s_1 v_{12} & s_2 v_{22} \end{pmatrix}$ , where  $v_1 = (v_{11}, v_{12})$ ,  $v_2 = (v_{21}, v_{22})$  are two vectors among the three primitive inner-normal vectors to the sides of the triangle  $\Delta$ , and  $s_1$  and  $s_2$  are the lattice lengths of the corresponding sides of  $\Delta$ .



Let  $f_{\Delta, unit}$  be the rational curve in  $\mathbb{V}_{\partial\Delta, \mathbb{P}^1}$  parameterized by the rational map

$$\phi_\Delta := \Psi_\Delta \circ \phi_\sigma : \theta \mapsto (\theta^{s_1 v_{11}} (\theta + 1)^{s_2 v_{21}}, \theta^{s_1 v_{12}} (\theta + 1)^{s_2 v_{22}}). \quad (3.3)$$

Let  $\mathbb{G}_\Delta$  be the 2-dimensional subtorus of  $\mathbb{T}_\Delta$  defined by the following embedding of  $(\mathbb{C} \setminus \{0\})^2$ ,

$$\Phi = \Phi_\Delta : (\mathbb{C} \setminus \{0\})^2 \rightarrow \mathbb{T}_\Delta, \quad \Phi_{ij}(\alpha, \beta) = \alpha^i \beta^j, \quad (i, j) \in \Delta \cap \mathbb{Z}^2. \quad (3.4)$$

Notice that any  $f \in \mathbb{V}_{\partial\Delta, \mathbb{P}^1}$  can be given by the following rational parametrization ([Shustin2, Lemma 3.5.]),

$$\theta \mapsto (\alpha \theta^{s_1 v_{11}} (\theta + 1)^{s_2 v_{21}}, \beta \theta^{s_1 v_{12}} (\theta + 1)^{s_2 v_{22}}). \quad (3.5)$$

That is,  $f = \Phi(\alpha^{-1}, \beta^{-1}) \cdot f_{\Delta, unit}$ , where  $\alpha, \beta \in (\mathbb{C} \setminus \{0\})^2$ . (Remember that the constant term is assumed to be always 1).

Therefore,  $\mathbb{V}_{\partial\Delta, \mathbb{P}^1}$  equals to  $f_{\Delta, unit} \mathbb{G}_\Delta$ .

□

3.3.  $\mathbb{V}_{\partial\mathcal{S}(\Delta)} \subset \mathbb{T}_\Delta$ . Now we consider a more general case. Let  $\mathcal{S}(\Delta)$  be a subdivision of  $\Delta$ .

**Definition 3.6.** Let  $\mathbb{V}_{\partial\mathcal{S}(\Delta)}$  denote the set of all  $f \in \mathbb{T}_\Delta$  such that  $f_{\Delta_i} \in \mathbb{V}_{\partial\Delta_i}$  for every  $\Delta_i \in Faces(\mathcal{S}(\Delta))$ , where  $f_{\Delta_i}$  is the restriction of  $f$  on  $\Delta_i$ .

**Theorem 3.7.**

- (1)  $\mathbb{V}_{\partial\mathcal{S}(\Delta)}$  is a closed subgroup of  $\mathbb{T}_\Delta$  up to a translation.
- (2) Its dimension equals to  $|Edges(\mathcal{S}(\Delta))| - |Faces(\mathcal{S}(\Delta))| + |Int(\mathcal{S}(\Delta)) \cap \mathbb{Z}^2|$ .

*Proof.* The proof is similar as the proof of Theorem 3.3.

Let  $f \in \mathbb{V}_{\partial\mathcal{S}(\Delta)}$ . First, assign a direction on each edge  $s$  of  $\mathcal{S}(\Delta)$  so that the univariate polynomial  $f_s$  corresponding to the restriction of  $f$  on the edge has the leading term and the constant term correspond to the head and the tail of the edge, respectively.

Second, assign a root  $\xi_s$  to  $f_s$  for every edge  $s \in Edges(\mathcal{S}(\Delta))$ .

Third, we construct one binomial equation for each face  $\Delta_i$  of  $\mathcal{S}(\Delta)$  as follows:

$$\prod_{s \in Edges(\mathcal{S}(\Delta))} (-\xi_s)^{\epsilon_s |s|} = 1, \quad (3.6)$$

where  $|s|$  is the lattice length of the edge  $s$  and

$$\epsilon_s = \begin{cases} 0 & \text{if } s \text{ is not a side of } \Delta_i \\ 1 & \text{if } s \text{ is a side of } \Delta_i \text{ and the direction of } s \\ & \text{is consistent with the counter-clock-wise orientation of } \Delta_i \\ -1 & \text{otherwise} \end{cases} \quad (3.7)$$

Then the restriction of  $f$  on the union of the edges of  $\mathcal{S}(\Delta)$  is uniquely determined by the system of binomial equations given above. It is straightforward to see that this system defines a closed subgroup  $\mathbb{G}$  of  $(\mathbb{C} \setminus \{0\})^m$  with coordinates  $(\xi_s)_{s \in \text{Edges}(\mathcal{S}(\Delta))}$ , where  $m = |\text{Edges}(\mathcal{S}(\Delta))|$  and this closed subgroup has the dimension  $|\text{Edges}(\mathcal{S}(\Delta))| - |\text{Faces}(\mathcal{S}(\Delta))|$ .

Also,  $f$  can take any non-zero complex number for every interior lattice point of  $\mathcal{S}(\Delta)$ .

Therefore we have an embedding of  $\mathbb{G} \times (\mathbb{C} \setminus \{0\})^{\text{Int}(\mathcal{S}(\Delta)) \cap \mathbb{Z}^2}$  into  $\mathbb{T}_\Delta$  which deduces the statements in the Theorem.  $\square$

*Remark 3.8.* The binomial equations (3.6) given in the proof of Theorem 3.7 easily provides a way of computing the number of components of  $\mathbb{V}_{\partial\mathcal{S}(\Delta)}$ , which is denoted by  $l(\partial\mathcal{S}(\Delta))$ . Namely, construct a matrix  $M_{\partial\mathcal{S}(\Delta)}$  whose rows and columns are indexed by  $\text{Faces}(\mathcal{S}(\Delta))$  and  $\text{Edges}(\mathcal{S}(\Delta))$  and the entries of  $M_{\partial\mathcal{S}(\Delta)}$  are written additively according to the binomials in (3.6), that is, the entries are  $\epsilon_s|s|$ 's. Then  $l(\partial\mathcal{S}(\Delta))$  equals to the greatest common divisor of the absolute values of  $m \times m$  minors of  $M_{\partial\mathcal{S}(\Delta)}$ , where  $m = |\text{Faces}(\mathcal{S}(\Delta))|$ . It can be computed easily from the Smith Normal Form of  $M_{\partial\mathcal{S}(\Delta)}$ . (Notice that this generalizes the  $l(\partial\Delta)$  in Theorem 3.3.)

3.4.  $\mathbb{V}_{\partial\mathcal{S}(\Delta), \mathbb{P}^1} \subset \mathbb{V}_{\partial\mathcal{S}(\Delta)}$ .

**Definition 3.9.** Let  $\mathbb{V}_{\partial\mathcal{S}(\Delta), \mathbb{P}^1}$  denote the set of all  $f \in \mathbb{V}_{\partial\mathcal{S}(\Delta)}$  such that  $f_{\Delta_i} \in \mathbb{V}_{\partial\Delta_i, \mathbb{P}^1}$  for every  $\Delta_i \in \text{Faces}(\mathcal{S}(\Delta))$ .

**Theorem 3.10.** Suppose that  $\mathcal{S}(\Delta)$  is triangular. Then the following hold true.

- (1)  $\mathbb{V}_{\partial\mathcal{S}(\Delta), \mathbb{P}^1}$  is a closed subgroup of the torus  $\mathbb{T}_\Delta$  up to a translation.
- (2) Its dimension equals to  $|\text{Vertices}(\mathcal{S}(\Delta))| - 1$ .

*Proof.* There are three steps to complete the proof.

First, we show that  $\mathbb{V} := \mathbb{V}_{\partial\mathcal{S}(\Delta), \mathbb{P}^1}$  is not empty.

Second, we construct a closed subgroup  $\mathbb{G}$  of  $\mathbb{T}_\Delta$  with dimension  $|\text{Vertices}(\mathcal{S}(\Delta))| - 1$ .

Last, we show that  $\mathbb{V}$  equals to the orbit  $f\mathbb{G}_{\mathcal{S}(\Delta)}$  of a point  $f \in \mathbb{V}$ .

*Step 1.* In the proof of Theorem 3.5, we saw that for a triangle  $\Delta$  any element in  $\mathbb{V}$  is uniquely determined by an element  $(\alpha, \beta) \in (\mathbb{C} \setminus \{0\})^2$  (Remember that we fix the base index at the origin and the constant term of an equation is always 1.) Let us denote this element by  $(\alpha, \beta) \cdot f_{\Delta, \text{unit}}$ . We extend this argument to the many-triangles case,  $\mathcal{S}(\Delta) : \Delta_1 \cup \dots \cup \Delta_m$ . We know that we can always find a  $(\alpha, \beta) \in (\mathbb{C} \setminus \{0\})^2$  such that  $(\alpha, \beta) \cdot f_{\Delta, \text{unit}}$  (up to a multiplication by a monomial) satisfies a given prescription on any two of the three sides of a triangle  $\Delta$ . That is, the coefficients of the equation  $(\alpha, \beta) \cdot f_{\Delta, \text{unit}}$  at the vertices of  $\Delta$  and the intersection points of the rational curve (defined by  $(\alpha, \beta) \cdot f_{\Delta, \text{unit}}$ ) with toric divisors corresponding to two of the three sides of  $\Delta$  are all prescribed. (for a proof, see [Shustin2, Lemma 3.5].)

Now we choose an orientation on the adjacency graph  $\mathcal{S}(\Delta)^*$  of  $\mathcal{S}(\Delta)$  which has no oriented cycle and no sink at vertices of 3-valency. (See 2.3 for details.)

It is clear that such oriented adjacency graph provides an algorithm to construct a point in  $\mathbb{V}$ . That is, we can choose a consistent collection of  $(\alpha, \beta)_{\Delta_i}$  for sub-polygons  $\Delta_i$  in  $\mathcal{S}(\Delta)$ .

*Step 2.* Choose a linear order,  $\Delta_1, \dots, \Delta_m$ , in the set of triangles in the subdivision  $\mathcal{S}(\Delta)$ . We choose one of the vertices of  $\Delta_1$  as the base index and assume that it is the origin. We order the set of inner edges in  $\mathcal{S}(\Delta)$  in the following way: Choose all inner edges belonging to  $\Delta_1$  (there are at most three such edges). Put them in an order. Then, choose all inner edges belonging to  $\Delta_2$  except the ones which may belong to  $\Delta_1$ . Add them in an order to the first set. In this way, we put a linear order in the set of all inner edges.

Each inner edge defines two binomial equations as follows: Let  $l = s_{ij}$  be the inner edge shared by  $\Delta_i$  and  $\Delta_j$ ,  $i < j$ . Let  $a = (a_1, a_2)$  be the lattice point of one of two ends of  $l$  and let  $v = (v_1, v_2)$  be the primitive



vector along  $l$  from  $a$ .

$$\gamma_i \alpha_i^{a_1} \beta_i^{a_2} - \gamma_j \alpha_j^{a_1} \beta_j^{a_2} = 0; \quad (3.8)$$

$$\alpha_i^{v_1} \beta_i^{v_2} - \alpha_j^{v_1} \beta_j^{v_2} = 0 \quad (3.9)$$

We collect the binomials for all inner edges and add one more binomial,  $\gamma_1 = 1$ . Let us denote this system by  $(\star)$ .

Then it is clear that this system is uniquely determined by the following system,

$$\gamma_1 = 1; \quad (3.10)$$

$$\alpha_i^{v_1} \beta_i^{v_2} \alpha_j^{-v_1} \beta_j^{-v_2} = 1(\star\star) \quad (3.11)$$

where the monomials in the left hand side of the equations are collected for all the inner edges.

Let  $\mathbb{G}$  be the closed subgroup in the torus  $(\mathbb{C} \setminus \{0\})^3 \times \cdots \times (\mathbb{C} \setminus \{0\})^3$  with coordinates

$(\alpha, \beta, \gamma) = ((\alpha_1, \beta_1, \gamma_1), \dots, (\alpha_m, \beta_m, \gamma_m))$  defined by the system  $(\star)$ .

Let  $M := M_{\partial\mathcal{S}(\Delta), \mathbb{P}^1}$  be the matrix corresponding to the monomials in the left hand side of equations in  $(\star\star)$  where the rows are indexed by the inner edges and the columns are indexed by  $(\alpha_i, \beta_i), i = 1, \dots, m$ . It is straightforward to see that the rows of  $M$  are linearly independent. Therefore,

$$\dim(\mathbb{G}) = 2|\text{Triangles}(\mathcal{S}(\Delta))| - |\text{IEEdges}(\mathcal{S}(\Delta))| \quad (3.12)$$

However, by the following Lemma 3.12,

$$\dim(\mathbb{G}) = |\text{Vertices}(\mathcal{S}(\Delta))| - 1. \quad (3.13)$$

Now we embed  $\mathbb{G}$  into  $\mathbb{T}_\Delta$  in the following way,

$$\Phi : \mathbb{G} \rightarrow \mathbb{T}_\Delta, \quad (3.14)$$

$$\Phi_{(w_1, w_2)}((\alpha_1, \beta_1, \gamma_1), \dots, (\alpha_m, \beta_m, \gamma_m)) = \gamma_k \alpha_k^{w_1} \beta_k^{w_2}, \quad (3.15)$$

where  $(w_1, w_2) \in \Delta_k \cap \mathbb{Z}^2, k = 1, \dots, m$ .

This map is well-defined because  $\mathbb{G}$  satisfies the system  $(\star)$ .

*Step 3.* Let us first show that  $f\mathbb{G} \subset \mathbb{V}$  for any  $f \in \mathbb{V}$ . (We don't distinguish  $\mathbb{G}$  from its image under the embedding  $\Phi$ . I believe that it will not cause a confusion.) Let  $(\alpha, \beta, \gamma) \in \mathbb{G}$ . The restriction of  $(\alpha, \beta, \gamma) \cdot f$  on  $\Delta_k$  is given by  $\gamma_k \cdot f_{\Delta_k}(\alpha_k x, \beta_k y)$ , which is a point in  $\mathbb{V}_{\Delta_k, \mathbb{P}^1}, k = 1, \dots, m$ . Thus, it is enough to show that the restrictions of  $(\alpha, \beta, \gamma) \cdot f$  on all sub-triangles coincide along the inner edges. It follows from the fact that  $\mathbb{G}$  satisfies the system  $(\star)$ .

Now we show that the other inclusion also holds.

Let  $h, h' \in \mathbb{V}$ . Then the restriction of  $h$  (resp.  $h'$ ) on  $\Delta_k$  has the following form up to a translation by a monomial  $x^{b_1} y^{b_2}$ ,

$$h_{\Delta_k}(x, y) = \gamma_k(\alpha_k, \beta_k) \cdot f_{\Delta_k, \text{unit}} = \gamma_k \cdot f_{\Delta_k, \text{unit}}(\alpha_k x, \beta_k y) \quad (3.16)$$

$$h'_{\Delta_k}(x, y) = \gamma'_k(\alpha'_k, \beta'_k) \cdot f_{\Delta_k, \text{unit}} = \gamma'_k \cdot f_{\Delta_k, \text{unit}}(\alpha'_k x, \beta'_k y), \quad (3.17)$$

for some  $(\alpha_k, \beta_k, \gamma_k)(\text{resp.}(\alpha'_k, \beta'_k, \gamma'_k)) \in (\mathbb{C} \setminus \{0\})^3, (b_1, b_2) \in \Delta \cap \mathbb{Z}^2$ , where  $k = 1, \dots, m$

Thus  $h'_{\Delta_k} = \gamma'_k \gamma_k^{-1} h_{\Delta_k}(\alpha'_k \alpha_k^{-1} x, \beta'_k \beta_k^{-1} y)$ .

That is,  $h'_{\Delta_k}$  is the restriction of  $(\alpha' \alpha^{-1}, \beta' \beta^{-1}, \gamma' \gamma^{-1}) \cdot h$  on  $\Delta_k$ . Therefore

$$h' = (\alpha' \alpha^{-1}, \beta' \beta^{-1}, \gamma' \gamma^{-1}) \cdot h. \quad (3.18)$$

We have completed the proof.  $\square$

*Remark 3.11.* As in Remark 3.8, we can compute the number of components of  $\mathbb{V}_{\partial\mathcal{S}(\Delta), \mathbb{P}^1}$ , easily from the matrix  $M_{\partial\mathcal{S}(\Delta), \mathbb{P}^1}$ . It equals to the greatest common divisor of all the absolute values of  $l \times l$  minors of  $M_{\partial\mathcal{S}(\Delta), \mathbb{P}^1}$ . But also it equals to the number of lattice points in the parallelepiped  $P = \{x_1 v_1 + \cdots + x_l v_l : 0 \leq x_i < 1, i = 1, \dots, l\}$ , where  $v_1, \dots, v_l$  are the row vectors of  $M_{\partial\mathcal{S}(\Delta), \mathbb{P}^1}$ .

**Lemma 3.12.** *If a subdivision  $\mathcal{S}(\Delta)$  of  $\Delta$  is triangular, then*

$$2|\text{Triangles}(\mathcal{S}(\Delta))| - |\text{IEEdges}(\mathcal{S}(\Delta))| = |\text{Vertices}(\mathcal{S}(\Delta))| - 1 \quad (3.19)$$

*Proof.* Let  $F$  be the number of the triangles,  $E'$  be the number of the edges on the boundary of  $\Delta$ , and let  $IE$  be the number of the inner edges in  $\mathcal{S}(\Delta)$ , respectively. Then,  $3 \cdot F = E' + 2 \cdot IE$ .

Since  $|Vertices| - |Edges| + |Faces| = 1$ ,

$$|Vertices(\mathcal{S}(\Delta))| - 1 = |Edges| - |Faces| = (E' + IE) - F = (3 \cdot F - IE) - F = 2 \cdot F - IE. \quad (3.20)$$

□

*Example 3.13.* Let  $\mathcal{S}(\Delta)$  be the following subdivision with 3 2-dimensional faces  $F_1, F_2, F_3$ .

Since  $\mathcal{S}(\Delta)$  has no interior lattice point,  $\mathbb{V}_{\partial\Delta} = \mathbb{V}_{\partial\Delta, \mathbb{P}^1}$ . Let us work on each case. Following as described in §3.3, we get the binomial system,

$$\begin{cases} \xi_1^{-2} \xi_2 \xi_3 &= 1 \\ \xi_2^{-1} \xi_4^4 \xi_5 &= 1 \\ \xi_3^{-1} \xi_5^{-1} \xi_6^2 &= 1 \end{cases} \quad (3.21)$$

The corresponding matrix  $M_{\partial\Delta}$  is:

$$M_{\partial\Delta} = \begin{pmatrix} & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ F_1 & -2 & 1 & 1 & 0 & 0 & 0 \\ F_2 & 0 & -1 & 0 & 4 & 1 & 0 \\ F_3 & 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix} \quad (3.22)$$

On the other hand, following as described in §3.4, we get the matrix  $M_{\partial\Delta, \mathbb{P}^1}$ :

$$M_{\partial\Delta, \mathbb{P}^1} = \begin{pmatrix} & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & \alpha_3 & \beta_3 \\ s_{12} & 1 & 2 & -1 & -2 & 0 & 0 \\ s_{13} & 1 & 0 & 0 & 0 & -1 & 0 \\ s_{23} & 0 & 0 & -1 & 2 & 1 & -2 \end{pmatrix} \quad (3.23)$$

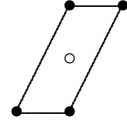
The Smith Normal Forms of  $M_{\partial\Delta}$  and  $M_{\partial\Delta, \mathbb{P}^1}$  coincide to each other as :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}. \quad (3.24)$$

Thus  $\mathbb{V}_{\partial\Delta} = \mathbb{V}_{\partial\Delta, \mathbb{P}^1}$  is a union of two translates of 3-dimensional subtorus of  $\mathbb{T}_\Delta$ .

3.5.  $\mathbb{V}_{\partial\Delta, \Sigma \mathbb{P}^1} \subset \mathbb{V}_{\partial\Delta}$ . Suppose that  $\Delta$  is a  $2m$ -gon with all opposite edges are parallel so that  $\Delta$  is the Minkowski sum of  $m$  segments  $s_1, \dots, s_m$ . Fix the base index and suppose that it is at the origin.

Let  $\mathcal{A} \subset \Delta_{\mathbb{Z}}$  be the set of lattice points in  $\Delta$  generated by  $v_1, \dots, v_m$ , where  $v_1, \dots, v_m$  are the primitive vectors along  $s_1, \dots, s_m$ , respectively. (The figure on the right shows that there are 5 lattice points in  $\Delta$ .  $\mathcal{A}$  consists of 4 of them.)



Let  $\mathbb{T}_{\mathcal{A}}$  denote the torus which parameterizes all  $f = f(x, y) \in \mathbb{P}(\mathcal{L}_\Delta)$  such that  $f$  has a non-zero coefficient for every monomial corresponding to a point in  $\mathcal{A}$ . (Remember that the constant term is always 1.)

**Definition 3.14.** Let  $\mathbb{V}_{\partial\Delta, \Sigma_{i=1}^m \mathbb{P}^1} \subset \mathbb{T}_{\mathcal{A}}$  denote the set of all  $f \in \mathbb{T}_{\mathcal{A}}$  such that  $f$  is the product of Laurent polynomials  $f_1, \dots, f_m$  whose Newton polygons are  $s_1, \dots, s_m$  respectively and each  $f_i$  is a pure power of a binomial for  $i = 1, \dots, m$ .

**Theorem 3.15.**  $\mathbb{V}_{\partial\Delta, \Sigma \mathbb{P}^1}$  is a subtorus of  $\mathbb{T}_{\mathcal{A}}$  with dimension equal to  $m = \frac{1}{2}|Edges(\Delta)|$  up to a translation.

*Proof.* Each  $f_i$  is of the form of  $(1 + \xi_i x^{v_i})^{|s_i|}$  for  $\xi_i \in (\mathbb{C} \setminus \{0\})$ , where  $|s_i|$  is the lattice length of  $s_i$ ,  $i = 1, \dots, m$ . Therefore the statement follows in straightforward. □

3.6.  $\mathbb{V}_{\partial\mathcal{S}(\Delta), nodal}$ . In this section, we consider the most general case which includes the previous ones as special cases.

Suppose that  $\mathcal{S}(\Delta) := \Delta_1 \cup \dots \cup \Delta_m$  is nodal, that is, every sub-polygon is either a triangle or a parallelogram. As in the previous section 3.5, we restrict  $\Delta$  to  $\mathcal{A}$ , where  $\mathcal{A} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_m$  and  $\mathcal{A}_i = \Delta_i \cap \mathbb{Z}^2$  if  $\Delta_i$  is a triangle and  $\mathcal{A}_i$  is defined as in the previous section if  $\Delta_i$  is a parallelogram.

**Definition 3.16.** Let  $\mathbb{V}_{\partial\mathcal{S}(\Delta), \text{nodal}}$  denote the set of all  $f \in \mathbb{T}_{\mathcal{A}}$  with the following properties,

- (▲) For every triangle  $\Delta_i$ ,  $f_{\Delta_i} \in \mathbb{V}_{\partial\Delta_i, \mathbb{P}^1}$ ;
- (■) For every parallelogram  $\Delta_j$ ,  $f_{\Delta_j} \in \mathbb{V}_{\partial\Delta_j, \mathbb{P}^1 + \mathbb{P}^1}$

**Theorem 3.17.**

- (1)  $\mathbb{V}_{\partial\mathcal{S}(\Delta), \text{nodal}}$  is a closed subgroup of the torus  $\mathbb{T}_{\mathcal{A}}$  up to a translation;
- (2) Its dimension equals to  $|\text{Vertices}(\mathcal{S}(\Delta))| - 1 - |\text{Parallelograms}(\mathcal{S}(\Delta))|$ .

*Proof.* The proof of the first statement follows from a simple adjustment of the proof of Theorem 3.10. Now the adjacency graph  $\mathcal{S}(\Delta)^*$  may have vertices of 4-valency. We choose a directed graph on  $\mathcal{S}(\Delta)^*$  which has no oriented cycle and no sink at vertices of 3 and 4-valency and also the edges in  $\mathcal{S}(\Delta)^*$  which are dual to the parallel sides of a parallelogram in  $\mathcal{S}(\Delta)$  are co-oriented. Using this directed graph, we can construct a point in  $\mathbb{V}_{\partial\mathcal{S}(\Delta), \text{nodal}}$ .

The construction of the closed subgroup  $\mathbb{G}$  is exactly same as the one given in the proof of Theorem 3.10.

The second statement follows from the following Lemma 3.18.  $\square$

**Lemma 3.18.** For a nodal subdivision  $\mathcal{S}(\Delta)$ , the following holds true:

$$2|\text{Triangles}| + 2|\text{Parallelograms}| - |\text{IEEdges}| = |\text{Vertices}| - 1 - |\text{Parallelograms}| \quad (3.25)$$

*Proof.* Let  $T := |\text{Triangles}|$ ,  $P := |\text{Parallelograms}|$ ,  $E := |\text{Edges}|$ ,  $IE := |\text{IEEdges}|$ ,  $V := |\text{Vertices}|$ , and  $F := |\text{Faces}|$ .

Then  $V - E + F = 1$ ,  $F = T + P$ , and  $3T + 4P = E + IE$ .

Thus  $V - 1 - P = E - T - 2P = (3T + 4P - IE) - T - 2P = 2T + 2P - IE$ .  $\square$

**Corollary 3.19.** In terms of the algebra  $\mathbb{A}$ , we can write as

$$\text{Trop}(\mathbb{V}) = l(\mathbb{V}) \cdot \text{Trop}(\mathbb{G}_{\circ}), \quad (3.26)$$

where  $\mathbb{V} = \mathbb{V}_{\partial\mathcal{S}(\Delta), \text{nodal}}$ ,  $l(\mathbb{V})$  is the number of components of  $\mathbb{V}$ , and  $\mathbb{G}_{\circ}$  is the identity component of the closed subgroup  $\mathbb{G} = \mathbb{G}_{\partial\mathcal{S}(\Delta), \text{nodal}}$  of  $\mathbb{T}_{\mathcal{A}}$ .

#### 4. INITIAL SCHEMES OF $\text{Sev}(\Delta, \delta)$

Now we are ready to describe the initial schemes of the very affine Severi variety  $\text{Sev}(\Delta, \delta)$ . Let  $\omega \in \mathbb{Z}^{\Delta_{\mathbb{Z}}}$  be an integral vector, which can be considered as an integral-valued function on  $\Delta_{\mathbb{Z}}$ , the set of lattice points on  $\Delta$ . We suppose that  $\omega((0, 0)) = 0$ , that is, the value of  $\omega$  at the origin (the base index of  $\Delta$ ) is always 0. As in §2.1, we may think of  $\omega$  as a function on monomials and we have the initial scheme  $\text{in}_{\omega}\text{Sev}(\Delta, \delta) \subset \mathbb{T}_{\Delta}$ . On the other hand, as in §2.3, we may consider the regular subdivision  $\Delta_{\omega}$  of  $\Delta$  constructed from  $\omega : \Delta_{\mathbb{Z}} \rightarrow \mathbb{Z}$ . Our main theorem 4.1 gives a description of the initial scheme  $\text{in}_{\omega}\text{Sev}(\Delta, \delta)$  in terms of the subdivision  $\Delta_{\omega}$  of  $\Delta$ .

**Theorem 4.1.** Let  $r$  denote the dimension of  $\text{Sev}(\Delta, \delta)$ .

- (1) If  $\text{rank}(\omega) > r$ , then  $\text{in}_{\omega}\text{Sev}(\Delta, \delta) = \emptyset$ .
- (2) If  $\text{rank}(\omega) = r$ , then  $\text{in}_{\omega}\text{Sev}(\Delta, \delta) \neq \emptyset$  if and only if  $\Delta_{\omega}$  is simple-nodal.
- (3) Furthermore, in the case that  $\text{rank}(\omega) = r$  and  $\text{in}_{\omega}\text{Sev}(\Delta, \delta) \neq \emptyset$ , we have an explicit description of  $\text{in}_{\omega}\text{Sev}(\Delta, \delta)$  as follows:
  - (a) As sets of closed points,  $\text{in}_{\omega}\text{Sev}(\Delta, \delta)$  equals to  $\mathbb{V} = \mathbb{V}_{\partial\Delta_{\omega}, \text{nodal}}$  upon a projection. Thus  $\text{in}_{\omega}\text{Sev}(\Delta, \delta)$  is the union of finitely many translates of a subtorus of dimension  $r$ .
  - (b) The number of such translates  $\mathbf{m}(\omega)$  (counted with multiplicity) equals to

$$l(\mathbb{V}) \cdot \prod \text{length}(\text{Edges}(\Delta_{\omega})), \quad (4.1)$$

where

- $l(\mathbb{V})$  is the number of components of  $\mathbb{V}$ ;
- $\prod \text{length}(\text{Edges}(\Delta_{\omega}))$  is the product of the lattice lengths of the edges which are representatives of each equivalence class in  $\text{Edges}(\Delta_{\omega})$ , where we define an equivalence relation as follows: let  $e \sim e'$  if  $e$  and  $e'$  are the parallel edges of a parallelogram in  $\Delta_{\omega}$  and extend it by transitivity.

*Remark 4.2.* As a corollary, if  $\text{in}_\omega \text{Sev}(\Delta, \delta) \neq \emptyset$  and the subdivision  $\Delta_\omega$  is either simple or nodal but not both, then  $\text{rank}(\omega) < r$ .

The proof of Theorem 4.1 uses Shustin's results on tropical limits and patchworking theory of nodal curves in the toric surface  $X_\Delta(\mathbb{K})$ . In the following section, we review them.

**4.1. Tropical limits and Patchworking Theory of plane curves over  $\mathbb{K}$ .** A tropical plane curve  $\tau_\omega$  is by definition the corner locus of the piecewise-linear function

$$\mathbb{R}^2 \rightarrow \mathbb{R}, \alpha \mapsto \max_{a \in \Delta_{\mathbb{Z}}} \{a \cdot \alpha + \omega(a)\}, \quad (4.2)$$

for some function  $\omega : \Delta_{\mathbb{Z}} \rightarrow \mathbb{Z}$ , where  $\Delta$  is called the *degree* of the tropical curve. This tropical curve is uniquely determined by  $cc(\omega)$ , the concave hull of  $\omega$ . For further details, refer to [IMS, Mikhalkin, Shustin2].

Let  $V_{\mathbb{K}}(f)$  be a curve defined by a Laurent polynomial over  $\mathbb{K}$ ,

$$f = \sum_{a \in \Delta_{\mathbb{Z}}} c_a(t) x^a \in \mathbb{K}[\mathbb{Z}^2], \quad c_a \in \mathbb{K} \setminus \{0\}, \quad (4.3)$$

where  $\mathbb{K}$  is the field of Puiseux series (defined in §2.1).

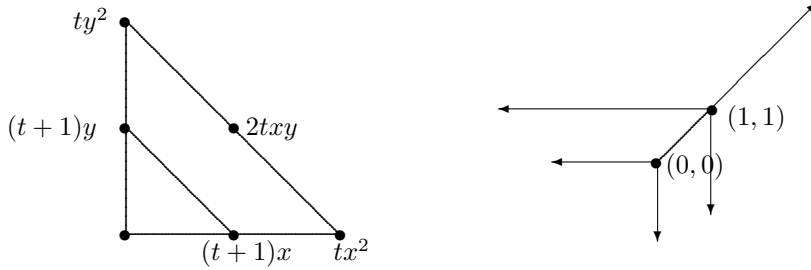
We will obtain combinatorial and geometric data of  $V_{\mathbb{K}}(f)$ , which is called the *tropical limit* of  $V_{\mathbb{K}}(f)$  (or of  $f$ ) (Definition 4.4).

The (support of) tropicalization of  $V_{\mathbb{K}}(f)$  is the tropical plane curve  $\tau_f := \tau_{\text{Val}_f}$ , where  $\text{Val}_f$  is the function  $\text{Val}_f : \Delta_{\mathbb{Z}} \rightarrow \mathbb{Z}, \quad a \mapsto \text{Val}(c_a(t))$ .

Then  $\tau_f$  is a 1-dimensional polyhedral complex, that is, it is a graph in  $\mathbb{R}^2$ , as it is shown in Example 4.3. The regular subdivision  $\Delta_f := \Delta_{\text{Val}_f}$  of  $\Delta$  is dual to  $\tau_f$  in the following sense:

- the components of  $\mathbb{R}^2 \setminus \tau_f$  are in 1-to-1 correspondence with  $\text{Vertices}(\Delta_f)$ .
- the edges of  $\tau_f$  are in 1-to-1 correspondence with  $\text{Edges}(\Delta_f)$  so that an edge of  $\tau_f$  is dual to an orthogonal edge of  $\Delta_f$ .
- the vertices of  $\tau_f$  are in 1-to-1 correspondence with the 2-dimensional faces of  $\Delta_f$  so that the valency of a vertex of  $\tau_f$  is equal to the number of sides of the dual face.

*Example 4.3.* Let  $f = (1 + x + y)(1 + tx + ty)$ . The picture below on the right is  $\tau_f$  and one on the left is the corresponding subdivision of the Newton polygon of  $f$ .



Next, we rewrite  $c_a(t) = \bar{c}_a t^{\text{Val}_f(a)} + \text{l.o.t.}$ ,  $\bar{c}_a \in (\mathbb{C} \setminus \{0\})$ ,  $a \in \Delta_{\mathbb{Z}}$  as follows:

$$c_a(t) = c_a^\circ t^{\nu_f(a)} + \text{l.o.t.}, \quad (4.4)$$

where  $\nu_f$  is the concave hull of  $\text{Val}_f$  (defined as in 2.3) and  $c_a^\circ$  is some complex number which is zero if  $\nu_f(a) > \text{Val}_f(a)$ .

Let  $\omega_i$  be the vertex of  $\tau_f$  corresponding to a face  $\Delta_i$  in  $\Delta_f : \Delta_1 \cup \dots \cup \Delta_m$ . Define

$$f_i := \sum_{a \in \Delta_{i,\mathbb{Z}}} c_a^\circ x^a, \quad (i = 1, \dots, m), \quad (4.5)$$

where  $\Delta_{i,\mathbb{Z}} = \Delta_i \cap \mathbb{Z}^2$ .

Therefore, from a curve  $V_{\mathbb{K}}(f)$  in  $(\mathbb{K} \setminus \{0\})^2$  with  $\text{Newton}(f) = \Delta$  we obtain a collection of curves  $V(f_i)$  in  $(\mathbb{C} \setminus \{0\})^2$  with  $\text{Newton}(f_i) = \Delta_i$ ,  $(i = 1, \dots, m)$ .

We collect all these combinatorial and geometric information by the following definition.

**Definition 4.4.** Let  $V_{\mathbb{K}}(f)$  be as above.  $(\nu_f, \Delta_f; f_1, \dots, f_m)$  is called the *tropical limit* of  $V_{\mathbb{K}}(f)$ .

Notice that the collection  $f_1, \dots, f_m$  uniquely determines a point in  $\mathbb{T}_{\mathcal{A}}$ , where  $\mathcal{A}$  is the subset of  $\Delta_{\mathbb{Z}}$  defined in §3.6. We denote this point by  $\{f_1, \dots, f_m\}$ .

The following proposition is almost identical with the statement which can be found in [Shustin2, §3.3], except one thing: his base-point-condition (i.e, passing a certain number of generic points) is replaced by the rank-condition (i.e.,  $\text{rank}(\omega) \geq \dim(\text{Sev}(\Delta, \delta))$ ). Our key observation about his argument is that he did not use the base-point-condition in his proof but the rank-condition. Therefore, we can use his proof as the proof of our statement.

**Proposition 4.5.** *Suppose  $\text{in}_{\omega}\text{Sev}(\Delta, \delta) \neq \emptyset$ . Then*

$$\text{rank}(\omega) \leq r \quad (4.6)$$

*Furthermore, if  $\text{rank}(\omega) = r$ , then we have a combinatorial and a geometric property as follows:*

- (combinatorial)  $\Delta_{\omega}$  is simple and nodal;
- (geometric) for any  $f \in \text{Sev}(\Delta, \delta)(\mathbb{K})$  with  $\omega = \text{Val}_f$ , the point  $\{f_1, \dots, f_m\}$  determined by the tropical limit of  $f$  is a point in  $\mathbb{V}_{\partial\Delta_{\omega}, \text{nodal}}$ .

Now we review on Shustin's patchworking theory.

In 1979-80, O. Viro suggested a patchworking construction for obtaining real nonsingular projective algebraic hypersurfaces with prescribed topology. This method was a major breakthrough in Hilbert's 16th problem. In the early 1990's, E. Shustin suggested to use the patchworking construction for tracing other properties of objects defined by polynomials, for example, prescribed singularities of algebraic hypersurfaces and many others. ([Shustin1, Shustin2]) He starts with a modified version of the patchworking construction, which allows one to keep singularities in the patchworking deformation. An important difference with respect to the original Viro method is that singularities are not stable in general, and thus one has to modify the Viro deformation and impose certain transversality conditions.

The following is a version of Shustin's enumerations of curves in toric surfaces [Shustin2, §3.7], summarized for the purposes of this paper.

Let  $\omega : \Delta_{\mathbb{Z}} \rightarrow \mathbb{Z}$  be an integral-valued function on  $\Delta_{\mathbb{Z}}$  such that  $\text{rank}(\omega) = r$  and  $\Delta_{\omega}$  is simple-triangular.

- (1) if we fix the coefficients  $c_b \in \mathbb{C} \setminus \{0\}$  for  $b \in \text{Vertices}(\Delta_{\omega})$ , then the number of  $F \in \mathbb{V} = \mathbb{V}_{\partial\Delta_{\omega}, \text{nodal}}$  with  $F(b) = c_b$  equals to  $\frac{\prod 2\text{area}(\text{Triangles})}{\prod \text{length}(\text{Edges}(\Delta_{\omega}))}$ , where the numerator stands for the product of twice the area of each triangle in  $\Delta_{\omega}$  and the denominator is defined as in Theorem 4.1.
- (2) if we fix the coefficients  $c_b(t) \in \mathbb{K} \setminus \{0\}$  for  $b \in \text{Vertices}(\Delta_{\omega})$  then the number of  $f \in \text{Sev}(\Delta, \delta)(\mathbb{K})$  with  $f(b) = c_b(t)$  equals to  $\prod 2\text{area}(\text{Triangles})$ . (In Shustin's notations in [Shustin2, §3.7], given  $c_b(t)$  for  $b \in \text{Vertices}(\Delta_{\omega})$ , the number of possible  $A$  (amoeba) is 1, the number of possible  $F$  (initial terms of coefficients of  $f$ ) is  $\frac{\prod 2\text{area}(\text{Triangles})}{\prod \text{length}(\text{Edges}(\Delta_{\omega}))}$ , and the number of possible  $R$  (deformation patterns) is  $\prod \text{length}(\text{Edges}(\Delta_{\omega}))$ . Thus the number of possible  $(A, F, R)$  equals to  $\prod 2\text{area}(\text{Triangles})$ , each of which gives rise to a unique  $f \in \text{Sev}(\Delta, \delta)(\mathbb{K})$ .)

*Remark 4.6.* In fact the enumerations above hold when the subdivision  $\Delta_{\omega}$  is simple-nodal, which was the case Shustin worked on. In this case, we replace  $\text{Vertices}(\Delta_{\omega})$  by a subset  $\mathcal{B}$  with  $|\mathcal{B}| = r + 1$  so that fixing coefficients for  $b \in \mathcal{B}$  uniquely determines the other coefficients for  $b \in \text{Vertices}(\Delta_{\omega}) \setminus \mathcal{B}$  for any  $F \in \mathbb{V}$ .

#### 4.2. Proof of Main Theorem 4.1.

*Proof.* From the first statement in Proposition 4.5, it follows that if  $\text{rank}(\omega) > r$ , then  $\text{in}_{\omega}\text{Sev}(\Delta, \delta) = \emptyset$ .

Now suppose that  $\text{rank}(\omega) = r$  and  $\text{in}_{\omega}\text{Sev}(\Delta, \delta) \neq \emptyset$ . Then the combinatorial property in the second statement in Proposition 4.5 implies that  $\Delta_{\omega}$  is simple-nodal.

Conversely, suppose that  $\text{rank}(\omega) = r$  and  $\Delta_{\omega}$  is simple-nodal. By Proposition 2.1, the set of closed points of  $\text{in}_{\omega}\text{Sev}(\Delta, \delta)$  is the set of  $(\bar{c}_a)_{a \in \Delta_{\mathbb{Z}}}$  such that there exists  $f \in \text{Sev}(\Delta, \delta)(\mathbb{K})$  of the following form

$$f(x) = \sum_{a \in \Delta_{\mathbb{Z}}} c_a(t) x^a; \quad (4.7)$$

$$c_a(t) = \bar{c}_a t^{\omega(a)} + \text{l.o.t.}, \quad \bar{c}_a \in \mathbb{C}^*, \quad (a \in \Delta_{\mathbb{Z}}). \quad (4.8)$$

Thus, Shustin's enumerations show that  $\text{in}_\omega \text{Sev}(\Delta, \delta) \neq \emptyset$ .

Now we prove the part (a) of the last statement.

Let us find the tropical limit of  $f$ : First, we *rewrite*  $f(x)$  as

$$f(x) = \sum_{a \in \Delta_{\mathbb{Z}}} c_a(t) x^a; \quad (4.9)$$

$$c_a(t) = c_a^\circ t^{\nu_f(a)} + l.o.t., \quad c_a^\circ \in \mathbb{C}, \quad (a \in \Delta_{\mathbb{Z}}), \quad (4.10)$$

where  $\nu_f$  is the concave hull of  $\omega = \text{Val}_f$ . Then,

$$c_a^\circ = \begin{cases} \bar{c}_a & \text{for } a \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}, \quad (4.11)$$

where  $\mathcal{A}$  is a subset of  $\Delta_{\mathbb{Z}}$  defined as in §3.6.

The geometric property in the second statement in Proposition 4.5 implies that  $(c^\circ)_{a \in \mathcal{A}}$  is a point in  $\mathbb{V} = \mathbb{V}_{\partial \Delta_\omega, \text{nodal}}$ . However, the projection,  $\pi : \mathbb{T}_\Delta \rightarrow \mathbb{T}_{\mathcal{A}}$  is injective on  $\text{in}_\omega \text{Sev}(\Delta, \delta)$  by Shustin's patchworking theory. Thus,  $\text{in}_\omega \text{Sev}(\Delta, \delta)$  is as a set equal to  $\mathbb{V}$  upon the projection.

Recall that  $\mathbb{V}$  is a closed subgroup  $\mathbb{G}$  of  $\mathbb{T}_{\mathcal{A}}$  up to a translation. Therefore  $\text{in}_\omega \text{Sev}(\Delta, \delta)$  is the union of finitely many translates of a subtorus  $\mathbb{G}_\circ^*$  of  $\mathbb{T}_\Delta$  with  $\pi(\mathbb{G}_\circ^*) = \mathbb{G}_\circ$ , where  $\mathbb{G}_\circ$  is the identity component of  $\mathbb{G}$ .

Finally, we prove the part (b) of the last statement.

Let  $\mathcal{B}$  be a subset of  $\text{Vertices}(\Delta_\omega)$  with the properties given in Remark 4.6, and let  $\mathbb{L}_{\mathcal{B}}$  be the  $n-1-r$  dimensional coordinate subspace of  $\mathbb{T}_\Delta$  defined by the equations  $x_b = 1$ ,  $b \in \mathcal{B}$ . Then Shustin's first enumeration (1) deduces the following:

$$\text{Trop}(\mathbb{V}) \pi_*(\text{Trop}(\mathbb{L}_{\mathcal{B}})) = \frac{\prod 2\text{area}(\text{Triangles})}{\widetilde{\prod} \text{length}(\text{Edges}(\Delta_\omega))}, \quad (4.12)$$

where  $\pi_* : \mathbb{R}^n \rightarrow \mathbb{R}^{|\mathcal{A}|}$  is the projection corresponding to  $\pi$ .

Moreover, from the second enumeration (2), we obtain the following:

$$\text{Trop}(\text{in}_\omega \text{Sev}(\Delta, \delta)) \text{Trop}(\mathbb{L}_{\mathcal{B}}) = \prod 2\text{area}(\text{Triangles}). \quad (4.13)$$

Thus,

$$\begin{aligned} m(\omega) \text{Trop}(\mathbb{G}_\circ^*) \text{Trop}(\mathbb{L}_{\mathcal{B}}) &= \text{Trop}(\text{in}_\omega \text{Sev}(\Delta, \delta)) \text{Trop}(\mathbb{L}_{\mathcal{B}}) \\ &= \prod 2\text{area}(\text{Triangles}) \\ &= \widetilde{\prod} \text{length}(\text{Edges}(\Delta_\omega)) \cdot \frac{\prod 2\text{area}(\text{Triangles})}{\widetilde{\prod} \text{length}(\text{Edges}(\Delta_\omega))} \\ &= \widetilde{\prod} \text{length}(\text{Edges}(\Delta_\omega)) \text{Trop}(\mathbb{V}) \pi_*(\text{Trop}(\mathbb{L}_{\mathcal{B}})) \\ &= l(\mathbb{V}) \widetilde{\prod} \text{length}(\text{Edges}(\Delta_\omega)) \text{Trop}(\mathbb{G}_\circ) \pi_*(\text{Trop}(\mathbb{L}_{\mathcal{B}})) \\ &= l(\mathbb{V}) \widetilde{\prod} \text{length}(\text{Edges}(\Delta_\omega)) \text{Trop}(\mathbb{G}_\circ^*) \text{Trop}(\mathbb{L}_{\mathcal{B}}) \end{aligned} \quad (4.14)$$

The last equality can be seen easily by considering the projection  $\pi_*$  as follows: by choosing a coordinate system,  $\text{Trop}(\mathbb{G}_\circ) \pi_*(\text{Trop}(\mathbb{L}_{\mathcal{B}}))$  is the determinant of a matrix  $(M_1 \mid M_2)$ , where  $M_1, M_2$  are found from lattice bases of  $\text{Trop}(\mathbb{G}_\circ)$  and  $\pi_*(\text{Trop}(\mathbb{L}_{\mathcal{B}}))$ , respectively. Then  $\text{Trop}(\mathbb{G}_\circ^*) \text{Trop}(\mathbb{L}_{\mathcal{B}})$  is the determinant of a matrix of the form  $\left( \begin{array}{c|c|c} M_1 & M_2 & 0 \\ * & 0 & I \end{array} \right)$ , where  $I$  is an identity matrix. Therefore  $\text{Trop}(\mathbb{G}_\circ) \pi_*(\text{Trop}(\mathbb{L}_{\mathcal{B}})) = \text{Trop}(\mathbb{G}_\circ^*) \text{Trop}(\mathbb{L}_{\mathcal{B}})$

Thus,

$$m(\omega) = l(\mathbb{V}) \widetilde{\prod} \text{length}(\text{Edges}(\Delta_\omega)) \quad (4.15)$$

□

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